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Thygesen, Uffe Høgsbro

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On Multi-Dissipative Dynamic Systems

Uffe Høgsbro Thygesen
Dept. of Mathematical Modeling
Technical University of Denmark
DK-2800 Lyngby
Denmark
E-mail: uht@imm.dtu.dk

Abstract

We consider deterministic dynamic systems with state space representations which are dissipative in the sense of Willems [10] with respect to several supply rates. This property is of interest in robustness analysis and in multi-objective control. We give conditions under which the convex cone of dissipated supply rates is closed. Furthermore we show convexity and continuity properties of the available storage and required supply as functions of the supply rate.

1 Introduction

Dynamic systems which are dissipative in the sense of Willems [10] appear in several areas of control theory. Roughly speaking, a system is dissipative if it is unable to produce a specified quantity, such as energy. The framework is applicable to large-scale systems and robustness problems because dissipativity is preserved under interconnections of systems and because dissipativity for autonomous systems implies stability. Indeed, the framework is a natural extension of Lyapunov theory to input/output systems. Although the notion of dissipativity is a quite general one, most attention has been given to two special cases: passive systems and systems with bounded \mathcal{L}_2 gain. See [10, 4, 9].

In this paper we consider deterministic dynamic systems which are dissipative with respect to *several* supply rates. Our interest in such multi-dissipative systems are due to two applications: It may be a *design objective* that a control system should be multi-dissipative, for instance that the closed loop has small gain and that the controller is passive. A multi-objective control problem in discrete time on a finite horizon is treated in [6]; this could be termed a problem of control for multi-dissipation. Our second application is that uncertain dynamic elements may be modeled as *multi-dissipative perturbations*. For instance, consider a mechan-

ical system containing two parasitics, each of which is passive and has small \mathcal{L}_2 gain. This results in a total of four dissipation properties which the parasitics satisfy together. Such information can be used to show robust stability and performance of the overall system as demonstrated in [11] for linear systems and quadratic supply rates.

Although much literature has been devoted to the topic of systems which are dissipative w.r.t *one* supply rate, it appears that simultaneous dissipation properties have not been studied. In this contribution we study properties of multi-dissipative systems which are related to topology and convexity. For instance, the set of supply rates w.r.t. which a system is dissipative is a closed convex cone, and for a fixed initial state, the available storage is a convex lower semi-continuous function of the supply rate (see below for definitions and exact statements). These properties are important both from an analytical and a computational point of view. We obtain the results using systems' theory and convex analysis but without theory for partial differential equations; in particular, we do not assume any form of continuity of the storage as function of the state.

The paper is organized as follows: In section 2 we summarize some definitions and properties associated with dissipative systems, mostly following [10]. Section 3 presents our new results for systems which are dissipative with respect to several supply rates while section 4 offers some conclusions.

Some statements in the paper are given without proof, which then can be found in [8] along with applications to robustness analysis and a generalization to stochastic systems.

2 Preliminaries

We consider dynamic systems Σ defined by ordinary differential equations in state-space:

$$\Sigma: \begin{aligned} \dot{x}(t) &= f(x(t), w(t)) \\ z(t) &= g(x(t), w(t)) \end{aligned} \quad (1)$$

Here, the system has input $w(t) \in \mathbb{W}$, output $z(t) \in \mathbb{Z}$ and state $x(t) \in \mathbb{X}$, and the spaces \mathbb{X} , \mathbb{W} and \mathbb{Z} are Euclidean. To avoid technicalities associated with existence and uniqueness of trajectories and finite escape times, we assume that $f: \mathbb{X} \times \mathbb{W} \rightarrow T\mathbb{X}$ is Lipschitz continuous, and that the input $w(\cdot)$ is restricted to the space \mathcal{W} of piecewise continuous locally bounded signals.

Associated with the system we have a locally bounded and measurable *supply rate* $s: \mathbb{W} \times \mathbb{Z} \rightarrow \mathbb{R}$ which describes a *flow* of some quantity into the system. When the initial state x_0 and the input $w(\cdot)$ is clear from the context we use the shorthand

$$s(t) := s(w(t), g(\phi(x_0, t, w(\cdot)), w(t))) .$$

We also assume that the read-out function $g: \mathbb{X} \times \mathbb{W} \rightarrow \mathbb{Z}$ is measurable and locally bounded. The essence in the technical assumptions on f , g , s and \mathcal{W} is that all signals are defined on the entire time interval $[0, \infty)$ and are measurable and locally bounded, and that the principle of optimality holds. Any other set of assumptions which guarantees this would suffice.

Our notion of dissipation is the original one of Willems [10]:

Definition 1: A dynamic system Σ is said to be *dissipative* with respect to the supply rate s (or, to dissipate s) if there exist a *storage function* $V: \mathbb{X} \rightarrow \mathbb{R}_+$ such that for all time intervals $[0, T]$, initial conditions x_0 and inputs $w \in \mathcal{W}$ the *dissipation inequality*

$$V(x(T)) \leq V(x(0)) + \int_0^T s(t) dt \quad (2)$$

holds. \square

We remark that James has proposed a slightly different definition in [5] where the storage function is required to be locally bounded. It is then possible to restrict attention to lower semi-continuous storage functions which are shown to be exactly the non-negative viscosity solutions in the sense of [2] to the differential formulation of the dissipation inequality

$$\forall w \in \mathbb{W}: V_x(x)f(x, w) \leq s(w, g(x, w)) \quad (3)$$

which must hold for all $x \in \mathbb{X}$. The two definitions coincide when the system is locally controllable; then all storage functions are continuous [4, 1].

A dissipative system will in general have many different storage functions for each supply rate, but two are of special interest. First we follow [10] and define the *available storage*

$$V_a(x) = \sup_{w(\cdot), T} \int_0^T -s(t) dt$$

where the integral is along the trajectory starting in x and corresponding to $w(\cdot)$. The available storage is finite everywhere if and only if the system is dissipative, in which case it is in itself a storage function and satisfies $V_a(x) \leq V(x)$ for any other storage function $V(\cdot)$. Furthermore, the available storage has infimum 0; the infimum needs not be attained.

Secondly, we define the required supply as the least possible supply which can bring the system from a state of minimal available storage to the desired terminal state. More precisely:

$$V_r(x) = \inf_{x(\cdot), w(\cdot), T} \int_0^T s(t) dt$$

where the trajectory $x(\cdot)$ must be consistent with $w(\cdot)$ and furthermore satisfy $V_a(x(0)) = 0$ and $x(T) = x$. When no such trajectory exists we define $V_r(x) = \infty$. The required supply satisfies $V_r(x) \geq V(x)$ for any storage function V which has been normalized so that $V(x) = 0$ whenever $V_a(x) = 0$. Furthermore, if $V_r(x)$ is finite everywhere (i.e. the system is dissipative, there exists at least one point of minimum available storage and any state is reachable from such a point) then $V_r(x)$ is in itself a storage function.

We remark that it is possible to give a more general definition which does not assume the existence of a point of minimum available storage; we shall not pursue this. Also, the reachability assumption will be used frequently in the following. In some situations where it does not hold it may be advantageous to redefine the state space of the system to contain exactly those states which are reachable.

Our definition of the required supply differs slightly from the one of Willems [10]: In this reference infimum is taken over trajectories which start in a fixed, specified point $x(0) = x^*$ with $V_a(x^*) = 0$. In contrast, we allow $x(0)$ to vary as long as $V_a(x(0))$ holds. We believe our definition is more suitable when multiple points of zero available storage exist, for instance several equilibrium points. A consequence of our definition is that any x which satisfies $V_a(x) = 0$ also satisfies $V_r(x) = 0$.

Sometimes we use the notation $V_a(x, s)$ and $V_r(x, s)$ to stress which supply rate we are referring to. In some situations it can be shown that the available storage and the required supply are viscosity solutions to differential dissipation *equations* corresponding to the inequality (3), provided that they are continuous [1].

3 Properties of multi-dissipative dynamic systems

We now consider a system Σ of the form (1) which is dissipative with respect to more than one supply rate. We investigate the set of supply rates which are dissipated by the system and we show several properties which are related to the convexity of this set.

In [10] it was noted that the storage functions for a dynamical system with respect to a single supply rate form a convex set, while [3] pointed out that the set of dissipated supply rates is a convex cone. In fact we immediately have the following:

Proposition 2: Let \mathcal{V} be a linear space of functions $\mathbb{X} \rightarrow \mathbb{R}$ and let \mathcal{S} be a linear space of supply rates $\mathbb{W} \times \mathbb{Z} \rightarrow \mathbb{R}$. Then those pairs (V, s) for which V is a storage function w.r.t. the supply rate s form a convex cone, i.e.

$$\{(V, s) \in \mathcal{V} \times \mathcal{S} \mid V \geq 0 \text{ and } (V, s) \text{ satisfy (2)}\}$$

is a convex cone. Furthermore this set is closed with respect to pointwise (in \mathbb{X}) convergence of storage functions V and local uniform convergence (over $\mathbb{W} \times \mathbb{Z}$) of supply rates s . \triangle

Regarding the last closedness statement, we restrict attention to the topology on \mathcal{S} corresponding to local uniform convergence over $\mathbb{W} \times \mathbb{Z}$; this mode of convergence appears to be most useful in applications. We remark that if $s_i \rightarrow s$ locally uniformly, then $\sup_{t \in [0, T]} |s_i(t) - s(t)| \rightarrow 0$ for any finite T and any trajectory since all signals are locally bounded functions of time.

The statement raises the question if the set of dissipated supply rates is closed. As an example, in \mathcal{L}_2 gain analysis one considers supply rates $s_\gamma(w, z) = \gamma^2 |w|^2 - |z|^2$ and define the \mathcal{L}_2 gain γ^* as the infimum over all numbers $\gamma > 0$ such that the system is dissipative w.r.t. s_γ . The question if the system is dissipative w.r.t. s_{γ^*} hence arises naturally. Notice that the closedness shown in proposition 2 does not answer such questions.

A first result in this direction is obtained with the notion of a cyclo-dissipative system:

Definition 3: The system Σ is cyclo-dissipative w.r.t. the supply rate s if

$$\int_0^T s(t) dt \geq 0$$

for any T and any pair $w(\cdot), x(\cdot)$ such that $x(0) = x(T)$. \square

This definition deviates slightly from the one in [4] where the inequality is required to hold only when $x(0) = x(T) = 0$; here, we have no reason to discriminate the state $x = 0$. A dissipative system is obviously cyclo-dissipative whereas

the converse implication does not hold in general [4]. We can now pose the statement:

Proposition 4: Assume that Σ is cyclo-dissipative w.r.t. s_i , $i \in \mathbb{N}$, and that $s_i \rightarrow s$ locally uniformly as $i \rightarrow \infty$. Then Σ is cyclo-dissipative w.r.t. s . \triangle

It follows that the set of cyclo-dissipated supply rates is a closed convex cone. An appealing conjecture is that the same statement holds if one replaces the word cyclo-dissipated with dissipated. This is not the case, however, as the following example demonstrates.

Example 5: Consider a scalar integrator, i.e. a system with state space $\mathbb{X} = \mathbb{R}$ and dynamics

$$\dot{x} = w, \quad z = x$$

and let the space \mathcal{S} of supply rates be the span of the two rates wz and z^3w . Consider a sequence of supply rates

$$s_i(w, z) = -2wz + \frac{4}{i}z^3w.$$

It is then easy to see that the system is dissipative w.r.t. s_i ; the available storage is

$$V_a(x, s_i) = -x^2 + \frac{1}{i}x^4 + \frac{i}{4}.$$

In fact the dissipation inequalities always hold with equality (in the terminology of [10] the system is *lossless* w.r.t. s_i). The supply rates s_i converge locally uniformly to $s(w, z) = -2wz$ and it is easy to see that the system does not dissipate s : For the dissipation inequality to be satisfied the storage function must necessarily be in the form $V(x) = -x^2 + K$ and no K exists such that V is non-negative. However, the system is cyclo-dissipative w.r.t. s in accordance with the previous result. \square

In order to get the desired result we need an additional assumption on the states of zero available storage:

Proposition 6: Let s_i , $i \in \mathbb{N}$, be a sequence of dissipated supply rates which converges locally uniformly to the supply rate s . Assume that the set of minimal available storage $\{x \mid V_a(x, s_i) = 0\}$ is independent of $i \in \mathbb{N}$ and non-empty, and that the entire state space \mathbb{X} is reachable from this set. Then the system is dissipative w.r.t. s . \triangle

Proof: (Sketch) Consider an arbitrary trajectory such that $V_a(x(0), s_i) = 0$ and define

$$J := \int_0^T s(t) dt$$

where $T > 0$ is arbitrary. Fix $\varepsilon > 0$ and choose i sufficiently large such that $|s_i(t) - s(t)| < \varepsilon$ for $t \in [0, T]$. It follows that

$$0 \leq \int_0^T s_i(t) dt \leq J + \varepsilon T.$$

Since $\varepsilon > 0$ was arbitrary we conclude that $J \geq 0$.

Now consider any continuation of the trajectory starting at time T in the state $x(T)$ and ending at time $T' > T$. Repeating the above argument we see that

$$\int_0^{T'} s(t) dt \geq 0$$

which in turn implies that

$$\int_T^{T'} -s(t) dt \leq J.$$

We conclude that $V_a(x(T), s) \leq J < \infty$ from which the claim follows. ■

The hypothesis that the set of zero initial storage is independent of i fails in example 5 above. In this example we have $V_a(x, s_i) = 0 \Leftrightarrow |x| = \sqrt{i/2}$.

Remark 7: The assumption that $\{x \mid V_a(x, s_i) = 0\}$ is independent of i may seem difficult to verify. In the case of linear systems with quadratic supply rates, an observability-type investigation will often yield that the state $x = 0$ is the only possible state of minimum available storage, in which case the assumption holds. This may be generalized to nonlinear systems with one compact and connected zero-input invariant set. However, if a nonlinear system has two isolated zero-input equilibria, the assumption is prone to fail. □

One can also derive closedness properties using theory for partial differential equations, rather than system theory, for instance following [2, sec. 6]. We point out that in comparison with this approach, proposition 6 has the strength of not imposing local boundedness, continuity, or other regularity requirements on the storage functions.

We now turn to the properties of the available storage and required supply, seen as functions of the supply rate.

Proposition 8: Let \mathbb{S} be a convex set of dissipated supply rates and let $x \in \mathbb{X}$ be fixed. Then $V_a(x, s)$ is a convex lower semi-continuous function of $s \in \mathbb{S}$. If furthermore the set $\{x \mid V_a(x, s) = 0\}$ is independent of $s \in \mathbb{S}$ and non-empty, and if the entire state space is reachable from this set, then $V_r(x, s)$ is a concave upper semi-continuous function of $s \in \mathbb{S}$. △

Proof: We show only that $V_a(x, \cdot)$ is lower semi-continuous: Let $s \in \mathbb{S}$ and let s_i be a sequence in \mathbb{S} which converges locally uniformly to s ; we must then show that

$$\liminf_{i \rightarrow \infty} V_a(x, s_i) \geq V_a(x, s).$$

Choose $\varepsilon > 0$ and let $x(\cdot)$ be a trajectory with $x(0) = x$ such

that

$$\int_0^T -s(t) dt \geq V_a(x, s) - \varepsilon.$$

Now choose i sufficiently large such that $|s_i(t) - s(t)| < \varepsilon/T$ on $[0, T]$; then

$$\int_0^T -s_i(t) dt \geq \int_0^T -s(t) dt - \varepsilon \geq V_a(x, s) - 2\varepsilon$$

which implies that $V_a(x, s_i) \geq V_a(x, s) - 2\varepsilon$ for i sufficiently large. Now let $\varepsilon \rightarrow 0$ to obtain the desired conclusion. ■

With this result in mind it is natural to ask if the available storage is also an upper semi-continuous function of the supply rate, and thus continuous. In general, the answer to this question is negative:

Example 9: Consider the autonomous system with state space $\mathbb{X} = \mathbb{R}$ and dynamics

$$\dot{x} = -x, \quad z = x.$$

Let a sequence of supply rates s_i be given by

$$s_i(z) = \begin{cases} -\frac{1}{i} & \text{if } 2^{-(i+1)/2} \leq z \leq 2^{-(i-1)/2} \\ 0 & \text{else.} \end{cases}$$

The space \mathcal{S} is the linear span of these s_i for $i \in \mathbb{N}$, and we take $\mathbb{S} = \mathcal{S}$. It is then straightforward to see that

$$V_a(x, s_i) = \log 2$$

for any $x \geq 1$. On the other hand, the supply rates s_i converge uniformly to the supply rate $s = 0$ for which $V_a(x, 0) = 0$. Hence the available storage $V_a(x, \cdot)$ is not an upper semi-continuous function of the supply rate. □

However, an example of particular interest is when the set \mathbb{S} of dissipated supply rates is a convex polytope, i.e. the convex hull of a finite collection of supply rates. In this situation upper semi-continuity follows from convexity [7, p. 84]:

Corollary 10: Take the same assumptions as in proposition 8 and assume in addition that \mathbb{S} is a convex polytope. Then $V_a(x, \cdot)$ and $V_r(x, \cdot)$ are continuous functions of $s \in \mathbb{S}$. □

We summarize and illustrate the discussion with the following simple example concerning \mathcal{L}_2 gain analysis of a scalar linear system.

Example 11: Consider the system

$$\dot{x} = -x + w, \quad z = x$$

and the two supply rates $s_1 = |w|^2$ and $s_2 = -|z|^2$ corresponding to an analysis of \mathcal{L}_2 gain from w to z . Let the space \mathcal{S} of supply rates be the span of s_1 and s_2 .

Since the system is linear and the supply rates are quadratic we know [10] that if the system is dissipative w.r.t. the rate $\lambda_1 s_1 + \lambda_2 s_2$ then there exist a quadratic storage function $V(x) = \alpha x^2$. The differential dissipation inequality then reduces to the linear matrix inequality

$$\begin{bmatrix} -2\alpha + \lambda_2 & \alpha \\ \alpha & -\lambda_1 \end{bmatrix} \leq 0 \quad .$$

The set of those $\alpha, \lambda_1, \lambda_2$, for which the linear matrix inequality holds, is a cone. Let us concentrate on the subcone for which $\lambda_2 > 0$. We may obtain a cross-section of this cone by fixing $\lambda_2 = 1$ and examine which values α and λ_1 result in a supply rate and a storage function which satisfy the dissipation inequality. A little manipulation yields that this set is characterized as

$$\alpha \geq \frac{1}{2}, \quad (2\alpha - 1)\lambda_1 \geq \alpha^2 \quad .$$

This set is depicted in figure 1. It has the structure which was predicted by the previous results: It is convex and closed as is its projection on the λ_1 -axis. Furthermore, the available storage and the required supply are continuous functions of λ_1 , convex and concave, respectively. In addition the set has the special feature of being unbounded since s_1 is sign definite. \square

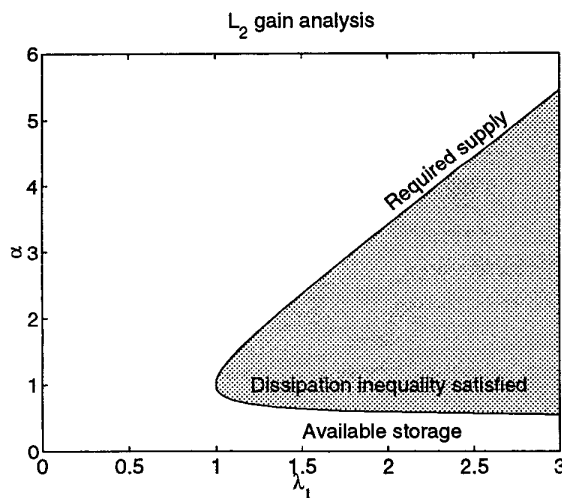


Figure 1: A cross section of the cone of supply rates and storage functions which satisfy the dissipation inequality.

4 Conclusion

One approach to robustness analysis of systems subject to dynamic perturbations is the following: First, using physical principles and/or experiments, we establish a number of

dissipation properties of the perturbations. Then we search for a *matching* dissipation property of the nominal system which will guarantee robustness [11, 8]. Finding the most suitable dissipation property of nominal system can be cast as an optimization problem

$$\min_{V,s} V(x_0, s)$$

subject to the constraint that V is a storage function for the nominal system with respect to s , and additional constraints on s (see [8] for details). This paper provides conditions on the nominal system under which this optimization problem is continuous in addition to being convex.

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